A simple proof of Sharkovsky's theorem rerevisited

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Abstract

Based on various strategies and three new doubling operators, we present several simple directed-graph proofs of the celebrated Sharkovsky's cycle coexistence theorem. A simple non-directed graph proof which is especially suitable for a calculus course right after the introduction of Intermediate Value Theorem is also given (in section 3).

1 Introduction

Throughout this note, I is a compact interval, and $f : I \to I$ is a continuous map. For each integer $n \ge 1$, let f^n be defined by: $f^1 = f$ and $f^n = f \circ f^{n-1}$ when $n \ge 2$. For x_0 in I, we call x_0 a periodic point of f with least period m or a period-m point of f if $f^m(x_0) = x_0$ and $f^i(x_0) \ne x_0$ when $0 < i < m$. If $f(x_0) = x_0$, then we call x_0 a fixed point of f.

For discrete dynamical systems defined by iterated interval maps on I, one of the most unexpected results is Sharkovsky's cycle coexistence theorem which states as follows:

Theorem (Sharkovsky[25, 26, 29, 31]) *Let the Sharkovsky's ordering of the natural numbers be defined (as suggested by Sharkovsky* [**31**]*) as follows:*

$$1 \prec 2 \prec 2^2 \prec 2^3 \prec \cdots \prec 2^n \prec \cdots \prec 9 \cdot 2^n \prec 7 \cdot 2^n \prec 5 \cdot 2^n \prec 3 \cdot 2^n \prec \cdots$$

$$\cdots \prec 9 \cdot 2^2 \prec 7 \cdot 2^2 \prec 5 \cdot 2^2 \prec 3 \cdot 2^2 \prec \cdots \prec 9 \cdot 2 \prec 7 \cdot 2 \prec 5 \cdot 2 \prec 3 \cdot 2 \prec \cdots \prec 9 \prec 7 \prec 5 \prec 3.$$

Then the following three statements hold:

(1) *If f has a period-m point and if $n \prec m$, then f also has a period-n point.*

(2) *For each positive integer n there exists a continuous map from I into itself that has a period-n point but has no period-m point for any m with $n \prec m$.*

(3) *There exists a continuous map from I into itself that has a period-2^i point for $i = 0, 1, 2, \dots$ but has no periodic point of any other period.*

It is clear that (1) is equivalent to the following three statements (cf. [**14, 15, 33**]):

(a) if f has a period-m point with $m = 3$ or 4, then f has a period-2 point;

(b) if f has a period-m point with $m \geq 3$ and odd, then f has a period-$(m+2)$ point; and

(c) if f has a period-m point with $m \geq 3$ and odd, then f has a period-6 point and a period-$(2m)$ point.

These three statements explain clearly why Sharkovsky's ordering is defined as it is. Note that in (a) we only need the special cases when $m = 3, 4$ (instead of all $m \geq 3$) which can be easily proved by discussing cases. Also, in (b) and (c), we don't require the existence of periodic points of all periods $\geq m+1$ and all *even* periods. Only the existence of period-$(m+2)$, period-6 and period-$(2m)$ points suffices. See section 11 for details.

In the past 30 years, there have been a number of papers dealing with Sharkovsky's theorem (see references), including the three papers [**13, 14, 15**] by the author and the "standard proof" developed in [**7, 8, 9, 20, 33**] and improved in [**1**] which, for the odd period cases, shows the existence of Štefan cycles first and then draws conclusions on (1) of Sharkovsky's theorem from the directed graphs of such cycles. On the other hand, in [**10**], Burns and Hasselblatt "select a salient sequence of orbit points and prove that this sequence 'spirals out' in essentially the same way as the Štefan cycles considered in the standard proof". Their proof and the standard's all use cycles of compact intervals with endpoints belonging to one and the same periodic orbit. By allowing these endpoints to be periodic or preperiodic points in different orbits, we have more cycles at our disposal and so can reach the goal more easily. Based on this tactic, we present in [**14**] a simple proof of (1) of Sharkovsky's theorem by going around the Štefan cycles. In this note, we shall present several strategies on how to prove (a), (b) and (c). We even confront the Štefan cycles with some quite straightforward arguments which achieve the same goal as the standard proof and yet surprisingly are just as simple as that in [**14**].

To make this note self-contained, we include some well-known preliminary results in section 2. In section 3, we give a non-directed graph proof of (a), (b) and (c) which uses the Intermediate Value Theorem in a very straightforward way. In sections 4, 5 and 6, we concentrate our attention on the point $\min P$ and/or the point $\max P$ in any given period-m orbit P. In section 7, we examine how the iterates of a given period-m point "jump" around a fixed point of f. In section 9, we investigate how the points in a given period-m orbit which lie on either side of a fixed point of f are mapped to the other side by f. In these two sections, we even prove, when $m \geq 5$ is odd, the existence of Štefan cycles of least period m without assuming the non-existence of smaller odd periods other than fixed points. In sections 8 and 10, we use the strategy in each previous section to treat the special case when f has a periodic point of odd period $m \geq 5$ but no periodic points of smaller odd periods other than fixed

points. The proof in each section is independent of the other. Some proofs of (a), (b), or (c) in these sections can be combined to give various complete proofs of (a), (b) and (c). Finally, in section 11, we present a complete proof of Sharkovsky's theorem in which the proof of (1) is different from those in **[14, 15]** while the examples for (2) and (3) which are constructed by a new doubling operator are completely new. In section 12, we introduce two more new doubling operators which, together with the classical one and the one in section 11, can be used in various combinations, as in section 11, to construct new examples for (2) and (3).

2 Preliminary results

To make this paper self-contained, we include the following well-known results.

Lemma 1. If $f^n(x_0) = x_0$, then the least period of x_0 with respect to f divides n.

Proof. Let m denote the least period of x_0 with respect to f and write $n = km + r$ with $0 \le r < m$. Then $x_0 = f^n(x_0) = f^{km+r}(x_0) = f^r(f^{km}(x_0)) = f^r(x_0)$. Since m is the smallest positive integer such that $f^m(x_0) = x_0$, we must have $r = 0$. Therefore, m divides n. ∎

Lemma 2. If J is a closed subinterval of I and $f(J) \supset J$, then f has a fixed point in J.

Proof. Write $J = [a, b]$. Since $f(J) \supset J \supset \{a, b\}$, there exist points p and q in $[a, b]$ such that $f(p) = a$ and $f(q) = b$. Let $g : I \longrightarrow R$ be a continuous map defined by $g(x) = f(x) - x$. Then $g(p) = f(p) - p = a - p \le 0$ and $g(q) = f(q) - q = b - q \ge 0$. By Intermediate Value Theorem, there is a point z between p and q such that $f(z) - z = g(z) = 0$. So, z is a fixed point of f in J. ∎

Lemma 3. *Let $k, m, n,$ and s be positive integers. Then the following statements hold:*

(1) *If y is a periodic point of f with least period m, then it is a periodic point of f^n with least period $m/(m, n)$, where (m, n) is the greatest common divisor of m and n.*

(2) *If y is a periodic point of f^n with least period k, then it is a periodic point of f with least period kn/s, where s divides n and is relatively prime to k.*

Proof. (1) Let t denote the least period of x_0 under f^n. Then m divides nt since $x_0 = (f^n)^t(x_0) = f^{nt}(x_0)$. Consequently, $\frac{m}{(m,n)}$ divides $\frac{n}{(m,n)} \cdot t$. Since $\frac{m}{(m,n)}$ and $\frac{n}{(m,n)}$ are coprime, $\frac{m}{(m,n)}$ divides t. On the other hand, $(f^n)^{(m/(m,n))}(x_0) = (f^m)^{(n/(m,n))}(x_0) = x_0$. So, t divides $\frac{m}{(m,n)}$. This shows that $t = \frac{m}{(m,n)}$.

(2) Since $x_0 = (f^n)^k(x_0) = f^{kn}(x_0)$, the least peirod of x_0 under f is $\frac{kn}{s}$ for some positive integer s. By (1), $(\frac{kn}{s})/((\frac{kn}{s}), n) = k$. So, $\frac{n}{s} = ((\frac{n}{s})k, n) = ((\frac{n}{s})k, (\frac{n}{s})s) = (\frac{n}{s})(k, s)$. This shows that s divides n and $(s, k) = 1$. ∎

Lemma 4. Let J and L be closed subintervals of I with $f(J) \supset L$. Then there exists a closed subinterval K of J such that $f(K) = L$.

Proof. Let $L = [a, b]$. Then since $\{a, b\} \subset L \subset f(J)$, there are two points p and q in J such that $f(p) = a$ and $f(q) = b$. If $p < q$, let $c = \max\{p \le x \le q : f(x) = a\}$ and let $d = \min\{c \le x \le q : f(x) = b\}$. If $p > q$, let $c = \max\{q \le x \le p \mid f(x) = b\}$ and let $d = \min\{c \le x \le p : f(x) = a\}$. In either case, let $K = [c, d]$. Then $f(K) = L$. ■

If there are closed subintervals $J_0, J_1, \cdots, J_{n-1}, J_n$ of I with $J_n = J_0$ such that $f(J_i) \supset J_{i+1}$ for $i = 0, 1, \cdots, n-1$, then we say that $J_0 J_1 \cdots J_{n-1} J_0$ is a *cycle of length* n. We need the following result which is useful for showing the existence of periodic points of some periods.

Lemma 5. *If $J_0 J_1 J_2 \cdots J_{n-1} J_0$ is a cycle of length n, then there exists a periodic point y of f such that $f^i(y)$ belongs to J_i for $i = 0, 1, \cdots, n-1$ and $f^n(y) = y$.*

Proof. Let $Q_n = J_0$. Since $f(J_{n-1}) \supset J_0 = Q_n$, there is, by Lemma 4, a closed subinterval Q_{n-1} of J_{n-1} such that $f(Q_{n-1}) = Q_n = J_0$. Continuing this process one by one, we obtain, for each $0 \le i \le n-1$, a closed subinterval Q_i of J_i such that $f(Q_i) = Q_{i+1}$. Consequently, $f^i(Q_0) = Q_i$ for all $0 \le i \le n$. In particular, $f^n(Q_0) = Q_n = J_0 \supset Q_0$. By Lemma 2, there is a point y in $Q_0 \subset J_0$ such that $f^n(y) = y$. Since $y \in Q_0$, we also obtain that $f^i(y) \in f^i(Q_0) = Q_i \subset J_i$ for all $0 \le i \le n-1$. ■

Remark. The point y obtained in Lemma 5 need not have least period n in general. However, by choosing appropriate cycles of length n, we can still get periodic points of least period n.

3 A non-directed graph proof of (a), (b) and (c)

The proof we present here is a slight improvement of the one in **[15]**. It is more direct and assumes no knowledge of dynamical systems theory whatsoever.

Let P be a period-m orbit of f with $m \ge 3$ and let $b = f^{m-1}(\min P)$. Then $f(b) = \min P < b$. If $f(x) < b$ on $[\min P, b]$, then, $(\min P \le)\ f^i(\min P) < b$ for all $i \ge 1$, contradicting the fact that $f^{m-1}(\min P) = b$. So, there is a point a in $[\min P, b]$ such that $f(a) \ge b$. Let z be a fixed point of f in $[a, b]$ and let v be a point in $[a, z]$ such that $f(v) = b$. Since $f^2(\min P) > \min P$ and $f^2(v) = \min P < v$, the point $y = \max\{\min P \le x \le v : f^2(x) = x\}$ exists. Furthermore, $f(x) > z$ on $[y, v]$ and $f^2(x) < x$ on $(y, v]$. Therefore, y is a period-2 point of f. (a) is proved.

For the proofs of (b) and (c), we assume that $m \ge 3$ is odd and note that $f(x) > z > x > f^2(x)$ on $(y, v]$. Since $f^{m+2}(y) = f(y) > y$ and $f^{m+2}(v) = f^2(v) = \min P < v$, the point $p_{m+2} = \min\{y \le x \le v : f^{m+2}(x) = x\}$ exists. Let k denote the least period of p_{m+2} with respect to f. Then $k > 1$ and k divides $m+2$ and so is odd. If $k < m+2$, then since $f^{k+2}(y) = f(y) > y$ and $f^{k+2}(p_{m+2}) = (f^2)(f^k(p_{m+2})) = f^2(p_{m+2}) < p_{m+2}$, there is a point w_{k+2} in (y, p_{m+2}) such that $f^{k+2}(w_{k+2}) = w_{k+2}$. Inductively, there exist points $y < \cdots < w_{m+2} < w_m < w_{m-2} < \cdots < w_{k+4} < w_{k+2} < p_{m+2} < v$ such that

$f^{k+2i}(w_{k+2i}) = w_{k+2i}$ for all $i \geq 1$, contradicting the fact that p_{m+2} is the smallest point in (y, v) which satisfies $f^{m+2}(x) = x$. Therefore, $k = m + 2$. This establishes (b).

We now prove (c). Let $z_0 = \min\{v \leq x \leq z : f^2(x) = x\}$. Then $f^2(x) < x$ and $f(x) > z$ on (v, z_0) and so also on (y, z_0). If $f^2(x) < z_0$ whenever $\min P \leq x < z_0$, then we have $\min P \leq f^{2i}(\min P) < z_0$ for all $i \geq 1$ which contradicts the fact that $(f^2)^{(m-1)/2}(\min P) = b > z_0$. Since $f^2(x) < x < z_0$ on (y, z_0), the point $d = \max\{\min P \leq x \leq y : f^2(x) = z_0\}$ exists and $f(x) > z \geq z_0 > f^2(x)$ on (d, y). Therefore, $f(x) > z \geq z_0 > f^2(x)$ on (d, z_0). Let $u_1 = \min\{d \leq x \leq v : f^2(x) = d\}$. Then $d < f^2(x) < z_0$ on (d, u_1). Let c_1 be any point in (d, u_1) such that $f^2(c_1) = c_1$. Let $u_2 = \min\{d \leq x \leq c_1 : f^2(x) = u_1\}$. Then $d < (f^2)^2(x) < z_0$ on (d, u_2). Let c_2 be any point in (d, u_2) such that $(f^2)^2(c_2) = c_2$. Inductively, we obtain points $d < \cdots < c_n < u_n < \cdots < c_2 < u_2 < c_1 < u_1 < z_0$ such that $d < (f^2)^n(x) < z_0$ on (d, u_n) and $(f^2)^n(c_n) = c_n$. Since $f(x) > z \geq z_0$ on (d, z_0), we have $f^i(c_n) < z_0 < f^j(c_n)$ for all even i and all odd j in $[0, 2n]$. So, each c_n is a period-$(2n)$ point of f. This proves (c).

Remarks. (1) It is well-known that if there exist a fixed point $\hat{z}$ of $h \in C^0(I, I)$, a point c in I and an integer $n \geq 2$ such that $h(c) < c < \hat{z} \leq h^n(c)$ then h has periodic points of all periods. In the above proof of (c), we have $f^2(v) < v < z < b = (f^2)^{(m+1)/2}(v)$. So, f^2 has periodic points of all periods. However, this fact does not guarantee the existence of period-$(2j)$ points for f for any odd $j \geq 3$. We need to do a little more work to ensure that as we did above. The following is another approach: Let m, v, z, b be defined as in the above proof. Let $g \in C^0(I, I)$ be defined by $g(x) = \max\{f(x), z\}$ if $x \leq z$ and $g(x) = \min\{f(x), z\}$ if $x \geq z$. Then $g([\min I, z]) \subset [z, \max I]$ and $g([z, \max I]) \subset [\min I, z]$. So, g has *no* periodic points of any *odd* periods ≥ 3. Since $m \geq 3$ is odd, for some $1 \leq i \leq m - 1$, both $f^i(b)$ and $f^{i+1}(b)$ lie on the same side of z. Let k be the *smallest* such i. Then the iterates $b, f(b), f^2(b), \cdots, f^k(b)$ are jumping alternately around z and since $f(v) = b$, so are the iterates $v, f(v), f^2(v), \cdots, f^{k+1}(v)$. Consequently, $g^i(v) = f^i(v)$ for all $0 \leq i \leq k + 1$. If k is odd, then $f^k(b) < z$ and $f^{k+1}(b) < z$ and so $g^{k+1}(b) = z$ and $g^2(v) = f^2(v) < v < z = g(z) = g^{k+2}(b) = g^{k+3}(v)$. If $k \geq 2$ is even, then $f^k(b) > z$ and $f^{k+1}(b) > z$ and so $g^{k+1}(b) = z$ and $g^2(v) = f^2(v) < v < z = g^{k+2}(v)$. In either case, we have $g^2(v) < v < z = (g^2)^n(v)$ for some $n \geq 2$. As noted above, this implies that g^2 has periodic points of all periods. In particular, g^2 has period-j points for all odd $j \geq 3$. So, g has either period-j points or period-$(2j)$ points for any odd $j \geq 3$. Since g has no periodic points of any odd periods ≥ 3, g has period-$(2j)$ points which are also period-$(2j)$ points of f for all odd $j \geq 3$. This establishes (c).

If for each odd integer $\ell \geq 3$, f has only finitely many period-ℓ points, then we have the following different approach: Let P be a period-m orbit of f such that $(\min P, \max P)$ contains no period-m orbits of f. Then by (b), $[\min P, \max P]$ contains a period-$(m + 2)$ orbit Q of f. Let h be the continuous map from I into itself defined by $h(x) = \min Q$ if $f(x) \leq \min Q$; $h(x) = \max Q$ if $f(x) \geq \max Q$; and $h(x) = f(x)$ elsewhere. Then h has no period-m points and by (b) h has no period-j points for any odd $3 \leq j \leq m - 2$. Since h has the period-$(m + 2)$ orbit Q, there exist a fixed point z of h and a point v such that $h(v) \in Q$ and $h^2(v) < v < z < h(v) = (h^2)^{(m+3)/2}(v)$. Consequently, h^2 has periodic points of all periods. Since h has no period-j points for any odd $3 \leq j \leq m$, h has period-$(2j)$ points for all odds $3 \leq j \leq m$ which are also periodic points of f with the same periods. In particular, f has period-6 and period-$(2m)$ points and so (c) is proved.

(2) The arguments in the above proofs of (a) and (b) can be used to give a simpler proof of the main result of Block in **[6]** on the stability of periodic orbits in Sharkovsky's theorem without resorting to the Štefan cycles. Indeed, assume that f has a period-2^n point. Let $F = f^{2^{n-2}}$. Then F has a period-4 orbit Q. Arguing as in the proof of (a), there exists a point v such that $\min Q = F^2(v) < v < F(v)$. Since $F^2(\min Q) > \min Q$, there is an open neighborhood U of f in $C^0(I, I)$ such that, for each g in U, the map $G = g^{2^{n-2}}$ satisfies $G^2(v) < v < G(v)$ and $G^2(\min Q) > \min Q$. Thus, the point $y = \max\{\min Q \le x \le v : G^2(x) = x\}$ is a period-2 point of G. Consequently, y is a period-2^{n-1} point of g. On the other hand, assume that f has a period-$(2^i \cdot m)$ point with $m \ge 3$ and odd and $i \ge 0$. Let $F = f^{2^i}$. Then F has period-m points. Arguing as in the proof of (a), there exist a period-2 point y of F and a point v with $y < v$ such that $F^{m+2}(y) > y$ and $F^{m+2}(v) = F^2(v) < v$ and F has no fixed points in $[y, v]$. So, there is an open neighborhood W of f in $C^0(I, I)$ such that, for each h in W, the map $H = h^{2^i}$ satisfies that $H^{m+2}(y) > y$, $H^{m+2}(v) < v$, and H has no fixed points in $[y, v]$. By Sharkovsky's theorem, h has period-$(2^i \cdot (m+2))$ points.

From now on, let P be a period-m orbit of f with $m \ge 3$. We shall present seven different directed graph (digraph for short) proofs of (a), (b) and (c). The first three proofs depend on the choice of the particular point $\min P$ and/or $\max P$. These particular choices simplify the proofs of (a), (b) or even (c) sometimes. In some cases, we shall need the following easy fact that if f has a period-3 point then f has periodic points of all periods. We shall also need the following easy result to show the existence of periodic points of all *even* periods.

Lemma 6. *Assume that there exist a point d and a fixed point z of f such that*

$$f^3(d) \le z \;\; \textit{and} \;\; f^2(d) < d < z < f(d), \quad \text{or} \quad f(d) < z < d < f^2(d) \;\; \textit{and} \;\; z \le f^3(d).$$

Then f has periodic points of all even periods. Furthermore, these periodic points can be chosen between z and $f(d)$ so that their iterates "spiral out" alternately around the fixed point z.

4 The first digraph proof of (a), (b) and (c)

Let $b = f^{m-1}(\min P)$. Then $f(b) = \min P$. If $f(x) < b$ for all $\min P \le x \le b$, then $\min P \le f^i(\min P) < b$ for all $i \ge 1$. This contradicts the fact that $b = f^{m-1}(\min P) < b$. Consequently, since $f(b) = \min P$, we have $f([\min P, b]) \supset [\min P, b]$. Let v be a point in $(\min P, b)$ such that $f(v) = b$ and let z be a fixed point of f in (v, b).

If for some point x in $[\min P, v]$, $f(x) \le z$, then we consider the cycle $[x, v][z, b][x, v]$. If for some x in $[\min P, v]$, $f(x) > z$ and $v < f^2(x)$, then we consider the cycle $[x, v][b : f(x)][x, v]$, where $[c : d] = [c, d]$ if $c < d$ and $[c : d] = [d, c]$ if $c > d$. If $f^2(x) \le v < z < f(x)$ for all x in $[\min P, v]$, let $p = \max([\min P, v] \cap P)$ $(\le v)$. Then $f^2([\min P, p] \cap P) \subset [\min P, p] \cap P$. Since f^2 is one-to-one on P, this implies that $f^2([\min P, p] \cap P) = [\min P, p] \cap P$ (and so m must be even). Consequently, $f^2([\min P, p]) \supset [\min P, p]$. In this case, we consider the cycle $[\min P, p] f([\min P, p]) [\min P, p]$. In either case, we obtain a period-2 point y of f such that $y < v < f(y)$. This proves (a).

We now prove (b). Let $m \ge 3$ be odd. Let k be the *largest even* integer in $[0, m-1]$ such that $f^k(\min P) < y$. Then $0 \le k \le m-2$ since $f^{m-1}(\min P) = b > y$. For $0 \le i \le m-2$, let $J_i = [f^i(\min P) : f^i(y)]$. If there is a *smallest odd* integer ℓ with $k < \ell \le m-2$ such that $f^\ell(\min P) < y$, then $y < f^i(\min P)$ for all $k < i < \ell$. We consider the cycle

$$J_k J_{k+1} J_{k+2} \cdots J_{\ell-1}[v, f(y)]([v, b])^{n+k-\ell-1} J_k$$

of length n for each $n \ge m+1$, where $([v, b])^{n+k-\ell-1}$ denotes the $n+k-\ell-1$ copies of $[v, b]$. If $y < f^j(\min P)$ for all odd integers $k < j < m-1$, we consider the cycle

$$J_k J_{k+1} J_{k+2} \cdots J_{m-2}([v, b])^{n-m+k+1} J_k$$

of length n for each $n \ge m+1$. In either case, we obtain a period-n point p_n in $[\min P, y]$ such that $p_n < y < f^i(p_n)$ for all $1 \le i \le n-1$. This confirms (b).

Here is a different proof. Let $m \ge 3$ be odd. Let $\hat{\ell}$ be the *smallest odd* integer in $[1, m]$ such that $f^{\hat{\ell}}(\min P) < y$. For $0 \le i \le \hat{\ell}-1$, let $I_i = [f^i(\min P) : f^i(y)]$. If $\hat{\ell} = m$, then since $f^{\hat{\ell}-1}(\min P) = b$, we consider, for each $n \ge m+1$, the cycle

$$I_0 I_1 I_2 \cdots I_{\hat{\ell}-2}([v, b])^{n-\hat{\ell}+1} I_0$$

of length n. If $1 \le \hat{\ell} \le m-2$, then we consider, for each $n \ge m+1$, the cycle

$$I_0 I_1 I_2 \cdots I_{\hat{\ell}-1}[v, f(y)]([v, b])^{n-\hat{\ell}-1} I_0$$

of length n. In either case, for each $n \ge m+1$, we have a period-n point q_n in $[\min P, y]$ such that $y < f^j(q_n)$ for all odd $1 \le j \le m$ and all $m+1 \le j \le n$. This also confirms (b). Note that the orbit of q_n need not be the same as that of p_n obtained above.

On the other hand, we can let $\tilde{\ell}$ be the *smallest odd* integer in $[1, m+2]$ such that $f^{\tilde{\ell}}(v) < y$. For $0 \le i \le \tilde{\ell}-1$, let $J_i = [f^i(y) : f^i(v)]$. If $\tilde{\ell} = m+2$, then $f^{\tilde{\ell}-1}(v) = f^{m+1}(v) = b$ and, for each $n \ge m+2$, we consider the cycle

$$J_0 J_1 J_2 \cdots J_{\tilde{\ell}-2}([v, b])^{n-\tilde{\ell}+1} J_0$$

of length n. If $\tilde{\ell} \le m$, then, for each $n \ge m+1$, we consider the cycle

$$J_0 J_1 J_2 \cdots J_{\tilde{\ell}-1}[v, f(y)]([v, b])^{n-\tilde{\ell}-1} J_0$$

of length n. In either case, we obtain, for each $n \ge m+2$, a period-n point of f whose orbit may be distinct from those of the points p_n and q_n obtained above. This confirms (b) too.

For the proof of (c), let $m \ge 3$ be odd and let $z_0 = \min\{v \le x \le z : f^2(x) = x\}$. Then we have $f(x) > z$ and $f^2(x) < x < z_0$ on $[v, z_0)$. If $f^2(x) < z_0$ on $[\min P, v]$, then $f^2(x) < z_0$ on $[\min P, z_0)$. Consequently, $(\min P \le)\, (f^2)^i(\min P) < z_0$ for each $i \ge 1$. Since $m \ge 3$ is odd, this contradicts the fact that $(f^2)^{(m-1)/2}(\min P) = b > z_0$. Hence $\max\{f^2(x) : \min P \le x \le v\} \ge z_0$. Let $I_0 = [\min P, v]$ and $I_1 = [v, z_0]$. Then $f^2(I_0) \cap f^2(I_1) \supset I_0 \cup I_1$. For each $n \ge 1$, by considering the cycle $I_1(I_0)^n I_1$ (with respect to f^2) of length $n+1$, we obtain a point w in $[v, z_0]$ such that $(f^2)^{n+1}(w) = w$ and $(f^2)^i(w) \in I_0$ for all $1 \le i \le n$. Thus $f^{2i}(w) < v < w < f(w)$ for all $1 \le i \le n$. In particular, the orbit of w under f contains at least $n+2$ distinct points. Consequently, w is a period-$(2n+2)$ point of f. Therefore, f has periodic points of all even periods ≥ 4. This establishes (c).

5 The second digraph proof of (a), (b) and (c)

Let a and b be points in $[\min P, \max P]$ such that $f(a) = \max P$ and $f(b) = \min P$. If $b < a$, let z be the *smallest* fixed point of f in (b, a). Since $f^k(\min P) = \max P > z$ for some $k \geq 1$, we have $\max\{f(x) : \min P \leq x \leq b\} > z$. By considering the cycles $[b, z]$ $([\min P, b])^i$ $[b, z]$, $i \geq 0$, we obtain periodic points of all periods for f. Now, let a and b be points in P such that $f(a) = \max P$ and $f(b) = \min P$. If $b < a$, then the above shows that f has periodic points of all periods and we are done. So, *for the rest of this section*, we assume that $a < b$.

Let v be the unique point in P such that $f(v) = b$. If $b < v$, then by considering the cycle $[b, v]([a, b])^i[b, v]$, $i \geq 1$, we obtain that f has periodic points of all periods ≥ 2. If $\min P < v < b$ and there is a fixed point $\tilde{z}$ of f in $[\min P, v]$, then from the cycles $[\tilde{z}, v]([v, b])^i[\tilde{z}, v]$, $i \geq 1$, we obtain that f has periodic points of all periods ≥ 2. So, *for the rest of this section*, we assume that $\min P < v < b$ and f has no fixed points in $[\min P, v]$. Since f^2 is one-to-one on P, the set $f^2([\min P, v] \cap P)$ contains at least as many points as the set $[\min P, v] \cap P$. Since $f^2(v) = \min P$, we have $f^2([\min P, v]) \supset [\min P, v]$. So, there is a point y in $[\min P, v]$ such that $f^2(y) = y$. Since f has no fixed points in $[\min P, v]$, y is a period-2 point of f such that $\min P < y < v < f(y)$. This establishes (a).

We now give a proof of (b). Let $m \geq 3$ be odd. Since $f^{m+2}([y, v]) \supset [f^{m+2}(y) : f^{m+2}(v)] = [\min P, f(y)] \supset [y, v]$, there is a point c in $[y, v]$ such that $f^{m+2}(c) = c$. If c has least period $m + 2$, we are done. Otherwise, let n be the least period of c under f. Then $n \geq 3$ and is a proper divisor of $m + 2$ which is odd and so $m + 2 \geq 3n > 2n$. Consequently, $m + 2 - n > (m + 2)/2$. Since $[f^n(y) : f^n(c)] = [c, f(y)] \supset [v, f(y)]$ and $f([v, b]) \supset [\min P, b]$, by considering the cycle

$$[y, c][f(y) : f(c)][f^2(y) : f^2(c)] \cdots [f^{n-1}(y) : f^{n-1}(c)][v, f(y)]([v, b])^{m+1-n}[y, c]$$

of length $m + 2$, we obtain a point d in $[y, c]$ such that $f^{m+2}(d) = d$, $f^n(d) \in [v, f(y)]$ and $f^i(d) \in [v, b]$ for all $n + 1 \leq i \leq m + 1$. Thus, $y < d < v < f^j(d)$ for all $n \leq j \leq m + 1$. Since there are at least $(m + 1) - n + 1$ $(> (m + 2)/2)$ *consecutive* iterates of d under f lying to the right of v and $y < d < v$, the least period of d under f is $m + 2$. This proves (b).

Now since $\min P = f(b) \leq a < b \leq \max P = f(a)$, there exist points $a < u < w < b$ such that $f(u) = b$ and $f(w) = a$. Thus, if $m \geq 3$ is odd, then P is also a period-m orbit of f^2 and $\min P = f^2(u) < u < w < f^2(w) = \max P$. By mimicking the proof in the first paragraph, we obtain that f^2 has a period-3 point. So, f has either a period-6 point or a period-3 point. In either case, f has a period-6 point. Now that we have shown the existence of period-6 points for f, to complete the proof of (c), it suffices to prove the existence of perood-$(2m)$ points for f. For this purpose, we present three different approaches as follows:

The first approach is independent of the existence of period-6 points of f and is very similar to the one as in the proof of (b) above. If $m = 3$, then f has periodic points of all periods and we are done. If $m \geq 5$ and odd, then since $f([f^m(y) : f^m(v)]) \supset [f^{m+1}(y) : f^{m+1}(v)] = [y, b] \supset [v, b]$, by considering the cycle

$$[y, v][f(y) : f(v)][f^2(y) : f^2(v)][f^3(y) : f^3(v)] \cdots [f^m(y) : f^m(v)]([v, b])^{m-1}[y, v]$$

of length $2m$, we obtain a period-$(2m)$ point p_{2m} of f such that $\min P < f^2(p_{2m}) < y < p_{2m} < v < f^k(p_{2m}) < b$ for all $m+1 \le k \le 2m-1$.

The second approach is based on the fact **[1, 7, 10]** that if f has a period-6 point then f has a period-6 point whose iterates are "jumping" alternately around a fixed point of f (such orbits are called *simple* in the literature). Indeed, if $\{x_1, x_2, \cdots, x_6\}$ with $x_1 < x_2 < \cdots < x_6$ is a period-6 orbit of f, let $s = \max\{1 \le i \le 6 : f(x_i) > x_i\}$. Then $f(x_s) \ge x_{s+1}$ and $f(x_{s+1}) \le x_s$ and there is a fixed point z of f in (x_s, x_{s+1}). If there is no $0 \le i \le 5$ such that $f^i(x_s)$ and $f^{i+1}(x_s)$ lie on the same side of z, then x_s is such a period-6 point of f. Otherwise, let r be the *smallest* integer in $[1,5]$ such that $f^r(x_s)$ and $f^{r+1}(x_s)$ lie on the same side of z. Then $1 \le r \le 4$. We now consider appropriate cycles of length 6 as follows:

If $r = 1$, the cycle $[x_s, z][x_{s+1}, f(x_s)]([x_s, z][z, x_{s+1}])^2[x_s, z]$,
if $r = 2$, the cycle $[x_s, z][z, f(x_s)][f^2(x_s), x_s]([z, x_{s+1}][x_s, z])^2$,
if $r = 3$, the cycle $[x_s, z][z, f(x_s)][f^2(x_s), z][x_{s+1}, f^3(x_s)][x_s, z][z, x_{s+1}][x_s, z]$,
if $r = 4$, the cycle $[x_s, z][z, f(x_s)][f^2(x_s), z][z, f^3(x_s)][f^4(x_s), x_s][z, x_{s+1}][x_s, z]$.

Or, we can choose points $x_s = x_0 < x_{-2} < x_{-4} < z < x_{-3} < x_{-1} < x_{s+1}$ such that $f(x_{-i}) = x_{-i+1}$, $1 \le i \le 4$, and consider the cycle

$$[x_{-5+r} : z][x_{-4+r} : z][x_{-3+r} : z] \cdots [x_s : z][f(x_s) : z][f^2(x_s) : z] \cdots [f^{r-1}(x_s) : z]\, L\, [x_{-5+r} : z]$$

of length 6, where $L = [f^r(x_s), x_s]$ if $f^r(x_s) < z$, and $L = [x_{s+1}, f^r(x_s)]$ otherwise. In either case, f has a period-6 point whose iterates are jumping alternately around the fixed point z of f. Consequently, the left three points form a period-3 orbit for f^2. By using the two adjcent compact intervals formed by these three period-3 points of f^2, we can find, for each $n \ge 1$, a period-n orbit Q_n of f^2 such that $Q_n \cup f(Q_n)$ is a period-$(2n)$ orbit of f. So, if f has a period-6 point then f has periodic points of all even periods, including period $2m$.

Here is the third approach: Suppose f has a period-6 point. Then f^2 has a period-3 orbit and so, for each *odd* $n \ge 5$, f^2 has a Štefan cycle of least period n. Indeed, without loss of generality, we may assume that $\{a_1, a_2, a_3\}$ is a period-3 orbit of f^2 with $f^2(a_3) = a_1 < a_2 = f^2(a_1) < a_3 = f^2(a_2)$. Let $\hat{z}$ be a fixed point of f^2 in (a_2, a_3). Since $f^2(a_3) < a_2 < \hat{z} < a_3 = f^2(a_2)$, there are points $u_0, u_{-1}, u_{-2}, \cdots$ such that $f^2(u_{-j}) = u_{-j+1}$, $j \ge 1$ and

$$a_2 = u_o < u_{-2} < \cdots < u_{-2i+2} < u_{-2i} < \cdots < \hat{z} < \cdots < u_{-2i+1} < u_{-2i+3} < \cdots < u_{-1} < a_3.$$

Let $J = [a_1, a_2]$. For each $k \ge 2$, by considering the cycle (with respect to f^2)

$$J[u_{-2k+2}, \hat{z}][\hat{z}, u_{-2k+3}][u_{-2k+4}, u_{-2k+2}][u_{-2k+3}, u_{-2k+5}] \cdots [u_{-2}, u_{-4}][u_{-3}, u_{-1}][a_2, u_{-2}][u_{-1}, a_3]J$$

of length $2k+1$, we obtain a point w_k in $J = [a_1, a_2]$ such that $(f^2)^{2k+1}(w_k) = w_k$ and

$$a_1 < w_k < a_2 < f^{4k-2}(w_k) < \cdots < f^6(w_k) < f^2(w_k) < \hat{z} < f^4(w_k) < \cdots < f^{4k}(w_k) < a_3.$$

So, w_k is a periodic point of f with least period $2k+1$ or $4k+2$. If w_k is a period-$(4k+2)$ point of f, then we are done. Otherwise, we have $f^{2k+4}(w_k) = f^3(w_k)$ and $f^{2k+6}(w_k) = f^5(w_k)$. Consequently, $f([f^2(w_k), f^4(w_k)]) \supset [f^3(w_k) : f^5(w_k)] = [f^{2k+4}(w_k) : f^{2k+6}(w_k)] \supset [f^2(w_k), f^4(w_k)]$. In particular, f has a fixed point in $[f^2(w_k), f^4(w_k)]$. Without loss of generality, we may assume that $\hat{z}$ is a fixed point of f. If k is odd, then $f^3(w_k) < f^2(w_k) <$

$\hat{z} < f^4(w_k)$ and we consider the cycle $[f^2(w_k), \hat{z}][f^2(w_k), \hat{z}][f^3(w_k), f^2(w_k)][f^2(w_k), \hat{z}]$. If k is even, then $f^5(w_k) < f^2(w_k) < \hat{z} < f^4(w_k) < f^3(w_k)$ and we consider the cycle $[f^2(w_k), \hat{z}][f^4(w_k), f^3(w_k)][\hat{z}, f^4(w_k)][f^2(w_k), \hat{z}]$. In either case, we obtain a period-3 point of f and so f has periodic points of all periods, including a period-$(4k+2)$ point. This shows that f has a period-$(2j)$ point for each odd integer $j \geq 3$, including period $2m$.

For simplicity, in the following five sections, we let $P = \{x_i : 1 \leq i \leq m\}$, with $x_1 < x_2 < \cdots < x_m$, be a period-m orbit of f with $m \geq 3$ and let s be an integer in $[1, m-1]$ such that $f(x_s) \geq x_{s+1}$ and $f(x_{s+1}) \leq x_s$. Let z be a fixed point of f in (x_s, x_{s+1}).

6 The third digraph proof of (a), (b) and (c)

We first show that if there exist a fixed point z^* of f, a point c and an integer $n \geq 2$ such that $f(c) < c < z^* < f^n(c)$ then there exist a fixed point $\tilde{z}$ of f and a point d such that $f(d) < d < \tilde{z} < f^2(d)$. Indeed, if $n = 2$, we are done. If $c < f^2(c) < f^3(c)$, then there is a fixed point $\tilde{z}$ of f in $(c, f^2(c))$ and we are done. If $f^2(c) < f(c)$, let $\hat{c} = f(c)$. If $f(c) < f^2(c) < c$, let $\hat{c}$ be a point in (c, z^*) such that $f(\hat{c}) = f^2(c)$. If $c < f^2(c) < z^*$ and $f^3(c) < f^2(c)$, let $\hat{c}$ be a point in $(f^2(c), z^*)$ such that $f(\hat{c}) = f^2(c)$. In either case, we have $f(\hat{c}) < \hat{c} < z^* < f^{n-1}(\hat{c})$. So, by induction, there is a fixed point $\tilde{z}$ of f and a point d such that $f(d) < d < \tilde{z} < f^2(d)$. Consequently, let u be a point in $(f(d), d)$ such that $\tilde{z} < f(u) < f(d)$ and let w be a point in (d, z) such that $f(w) = u$. Let $I_0 = [u, d]$ and $I_1 = [w, z]$. Then $I_0 \cap I_1 = \emptyset$ and $f(I_0) \cap f(I_1) \supset I_0 \cup I_1$. By considering the cycles $I_0(I_1)^n I_0$, $i \geq 0$, f has periodic points of all periods. Now, let a and b be two points in P such that $f(a) = x_m$ and $f(b) = x_1$. If $b < a$, then $x_1 = f(b) < b < z < a \leq x_m = f^k(b)$ for some $k \geq 2$. The above implies that f has periodic points of all periods and we are done. So, *for the rest of this section*, we assume that $a \leq x_s < z < x_{s+1} \leq b$.

It is clear that one side of z contains at least as many points of P as the other side. *We may assume that it is the right side*, (if it is the left side, the proof is similar). If the right side contains as many points of P as the left side, then, since $a \leq x_s < z < x_{s+1} \leq b$, we have $f([x_{s+1}, x_m]) \supset [x_1, x_s]$ and $f([x_1, x_s]) \supset [x_{s+1}, x_m]$. If the right side contains more points of P than the left side, then we have $f([x_{s+1}, x_m]) \supset [x_1, x_{s+1}] \supset [x_1, z]$ and $f([x_1, z]) \supset f([a, z]) \supset [x_{s+1}, x_m]$. In either case, f has a period-2 point. This proves (a).

For the proof of (b) and (c), we assume that $m \geq 3$ is odd and the right side of z contains more points of P than the left side. So, we have $1 \leq s < m/2$ and $f([x_{s+1}, x_m]) \supset [x_1, x_{s+1}]$. If $s = 1$, then $f(x_2) = x_1$ and $f(x_1) = f(a) = x_m$, and so $f([x_1, x_2]) \supset [x_1, x_m]$. By considering the cycles $[x_2, x_m]([x_1, x_2])^n[x_2, x_m]$, $n \geq 1$, we have periodic points of all periods ≥ 2 for f and we are done. So, *for the rest of this section*, suppose $2 \leq s < m/2$ (and so $m \geq 5$). We may also suppose f has no period-ℓ points with ℓ odd and $1 < \ell < m$.

Let $L = [x_{s+1}, x_m]$. Then $f(L) \supset [x_1, x_{s+1}] \supset [a, x_{s+1}]$ and so $f^2(L) \supset [x_s, x_m] \supset L$. Since P is also a period-m orbit of f^2, $f^2(P \cap [x_{s+1-k}, x_m]) \not\subset [x_{s+1-k}, x_m]$ for each $1 \leq k \leq s-1$. Therefore, since $x_m \in f^2(L)$, $f^2([x_{s+1-k}, x_m]) \supset [x_{s-k}, x_m]$ for all $1 \leq k \leq s-1$.

Consequently, since $f^2(L) \supset [x_s, x_m]$, we have $f^{2s}(L) \supset [x_1, x_m]$. By considering the cycle $Lf(L)f^2(L)f^3(L)\cdots f^{2s-1}(L)[x_1, z]L$ of length $2s+1\,(\le m)$, we obtain a period-ℓ point of f with ℓ odd and $1 < \ell \le 2s+1 \le m$. Since f has no periodic points of odd periods ℓ with $1 < \ell < m$, this forces $2s+1 = m$. If $x_1 < a \le x_s$, or $f^2([x_{s+1-k}, x_m]) \supset [x_{s-k-1}, x_m]$ for some $0 \le k \le s-2$, then since $f^2([x_1, x_m]) \supset [x_1, x_m]$, we have $f^{2(s-1)}(L) \supset [x_1, x_m]$. So, by considering the cycle $Lf(L)f^2(L)f^3(L)\cdots f^{2s-3}(L)[a, z]L$ of length $2s-1 = m-2$, we obtain a period-ℓ point of f with ℓ odd and $1 < \ell < m$. This is a contradiction. Therefore, $a = x_1$ and $f^2([x_{s+1-k}, x_m]) \not\supset [x_{s-k-1}, x_m]$ for all $0 \le k \le s-2$. Since $f^2([x_{s+1-k}, x_m]) \supset [x_{s-k}, x_m]$ for all $1 \le k \le s-1$, we obtain that $P \cap f^2([x_{s+1-k}, x_m]) = P \cap [x_{s-k}, x_m]$ for all $0 \le k \le s-1$.

Since $x_{s+1} \in [x_1, x_{s+1}] \subset f(L)$, we have $f(x_{s+1}) \in P \cap f^2(L) = P \cap [x_s, x_m]$. In particular, $f(x_{s+1}) \ge x_s$. But since $f(x_{s+1}) \le x_s$, this implies $f(x_{s+1}) = x_s$. If $f^2(x_s) \in [x_s, x_m]$, then since $P \cap f^2(L) = P \cap [x_s, x_m]$, we have $f^2(P \cap [x_s, x_m]) \subset \{f^2(x_s)\} \cup \big(P \cap f^2(L)\big) \subset [x_s, x_m]$ which is a contradiction. So, $f^2(x_s) < x_s$. Since $P \cap f^2([x_s, x_m]) = P \cap [x_{s-1}, x_m]$, we have $f^2(x_s) = x_{s-1}$. Inductively, we obtain that $f^2(x_{s+1-k}) = x_{s-k}$ for each $1 \le k \le s-1$. In particular, $x_{s-k} = f^{2k}(x_s)$ for each $1 \le k \le s-1$ and $x_m = f(a) = f(x_1) = f(f^{2(s-1)}(x_s)) = f^{m-2}(x_s)$. Since $f(x_{s+1}) = x_s$, this shows that $f^j(x_{s+1}) < z < f^i(x_{s+1}) < f^{m-1}(x_{s+1}) = x_m$ for all odd $1 \le j < m$ and all even $0 \le i < m-1$. For each $n > m$, by considering the cycle

$$[z, x_{s+1}][z : f(x_{s+1})][z : f^2(x_{s+1})]\cdots[z : f^{m-2}(x_{s+1})][x_{m-1}, x_m]([x_s, x_{s+1}])^{n-m}[z, x_{s+1}]$$

of length n, we obtain a period-n point of f. This establishes (b). Furthermore, let $U = [z, f^{m-3}(x_{s+1})]$, $V = [f^{m-2}(x_{s+1}), z]$ and $W = [f^{m-3}(x_{s+1}), f^{m-1}(x_{s+1})]$. Since $f(U) \supset V$, $f(V) \supset W$ and $z \in f(W)$, it is easy to see that (c) follows from Lemma 6. Note that P is actually a Štefan cycle. However, we don't need this fact for the proof of (b) and (c). See Case 1 in section 8 for details.

7 The fourth digraph proof of (a), (b) and (c)

We now consider how the iterates $x_s, f(x_s), f^2(x_s), \cdots, f^{m-1}(x_s)$ "jump" around z.

If for all integers k such that $1 \le k \le m-1$ the points $f^k(x_s)$ and $f^{k+1}(x_s)$ lie on opposite sides of z, then, since $f(x_s) > z > x_s$, m is even and, $f^i(x_s) < z < f^j(x_s)$ for all even i and all odd j in $[0, m-1]$. This implies that $f([x_1, x_s] \cap P) = [x_{s+1}, x_m] \cap P$ and $f([x_{s+1}, x_m] \cap P) = [x_1, x_s] \cap P$. In particular, $f([x_1, x_s]) \supset [x_{s+1}, x_m]$ and $f([x_{s+1}, x_m]) \supset [x_1, x_s]$. By considering the cycle $[x_1, x_s][x_{s+1}, x_m][x_1, x_s]$, we obtain a period-2 point of f.

Now assume that there is a *smallest* integer r in $[1, m-1]$ such that the points $f^r(x_s)$ and $f^{r+1}(x_s)$ lie on the same side of z (and so the iterates $x_s, f(x_s), f^2(x_s), \cdots, f^r(x_s)$ are jumping around z alternately and this includes the case when m is odd). We may also assume that, *for the rest of this section*, both $f^r(x_s)$ and $f^{r+1}(x_s)$ lie on the right side of z (if they both lie on the left side of z, the proof is similar). Then r is odd and $1 \le r \le m-2$. Since $f(x_{s+1}) \le x_s < x_{s+1}$, we have $f^r(x_s) > x_{s+1}$. Let k be the *smallest* integer in $[1, r]$ such that $f^k(x_s) \ge f^r(x_s)$. Since the iterates $x_s, f(x_s), f^2(x_s), \cdots,\ f^k(x_s)$ are "jumping" alternately around z, we see that k is odd. If $k = 1$, then for all $i \ge 2$, by considering the cycles

$[x_{s+1}, f^r(x_s)]([x_s, x_{s+1}])^{i-1}[x_{s+1}, f^r(x_s)]$ of length i, we see that f has periodic points of all periods ≥ 2. So, suppose $k \ge 3$. Thus, $f^{k-1}(x_s) < z < f^{k-2}(x_s) < f^r(x_s) \le f^k(x_s)$. Let $U = [z, f^{k-2}(x_s)]$, $V = [f^{k-1}(x_s), z]$ and $W = [f^{k-2}(x_s), f^r(x_s)]$. Then $f(U) \supset V$, $f(V) \supset W$ and $z \in f(W)$. By Lemma 6, this proves (c) and also (a).

On the other hand, let $x_0, x_{-1}, x_{-2}, \cdots$ be points in $[x_s, x_{s+1}]$ such that

$$x_s = x_0 < x_{-2} < x_{-4} < \cdots < z < \cdots < x_{-3} < x_{-1} < x_{s+1}$$

and $f(x_{-i}) = x_{-i+1}$ for all $i = 1, 2, \cdots$. For each *odd* integer $n \ge m$, let $n_0 = n - 1 - r$. Note that $n_0 + r = n - 1$ is even. By considering the cycle

$$[f^{r-2}(x_s) : f^r(x_s)][z, x_{-r_0}][x_{-r_0+1}, z][x_{-r_0}, x_{-r_0+2}][x_{-r_0+3}, x_{-r_0+1}] \cdots [x_{-3}, x_{-1}][x_s, x_{-2}]$$
$$[x_{-1}, f(x_s)][f^2(x_s), x_s][f(x_s) : f^3(x_s)][f^2(x_s) : f^4(x_s)] \cdots [f^{r-2}(x_s) : f^r(x_s)]$$

of length n, we obtain a period-n point p_n in $[f^{r-2}(x_s) : f^r(x_s)]$ whose iterates $f^i(p_n)$, $1 \le i \le n$, are jumping alternately around z. This confirms (b). We can also consider the cycle

$$[x_{s+1}, f^r(x_s)]([z, x_{s+1}][x_s, z])^{(n-r)/2}[z, f(x_s)][z : f^2(x_s)] \cdots [z : f^{r-1}(x_s)][x_{s+1}, f^r(x_s)]$$

of length n to obtain a period-n point q_n in $[x_{s+1}, f^r(x_s)]$ such that the iterates $f^i(q_n)$, $1 \le i \le n$, are jumping alternately around z. This also proves (b). Note that the orbit of q_n need not be the same as that of the above point p_n.

With a little more work than the above proof of (b), we can even show the existence of all Štefan cycles of odd periods $n \ge m$ *without assuming the non-existence of odd periods ℓ with* $1 < \ell < m$. Indeed, since we suppose r is odd, $1 \le r \le m - 2$. Let u be a point in $[z, x_{s+1}]$ such that $f(u) = x_s$. By considering the cycle

$$[z, u][f(u), z][z, f^2(u)][z : f^3(u)][z : f^4(u)] \cdots [z : f^r(u)][u : f^{r+1}(u)][z, u]$$

of length $r+2 \le m$, we obtain a period-$(r+2)$ point w of f in $[z, u]$ such that the iterates $f^j(w)$, $0 \le j \le r + 1$, are jumping alternately around z and $z < w < f^{r+1}(w)$. For $0 \le i \le r + 1$, let $J_i = [z : f^i(w)]$. If the orbit of w with respect to f is not a Štefan cycle, then, for some $0 \le i_2 < i_1 \le r + 1$ with $i_1 - i_2$ even, we have $[f^{i_1}(w) : z] \subset [f^{i_2}(w) : z]$. If $i_2 = 0$, then $i_1 < r + 1$ and we consider the cycle $J_{i_1} J_{i_1+1} J_{i_1+2} \cdots J_r [w, f^{r+1}(w)] J_{i_1}$ of length $r + 2 - i_1$. Otherwise, we consoder the cycle $J_0 J_1 \cdots J_{i_2-1} J_{i_1} J_{i_1+1} \cdots \ J_r \ [w, f^{r+1}(w)] J_0$ if $i_1 < r + 1$ or the cycle $J_0 J_1 J_2 \cdots J_{i_2-1}[w, f^{r+1}(w)] J_0$ if $i_1 = r+1$ of odd length $r+2-i_1+i_2$. In either case, we obtain a periodic point $\hat{w}$ of f with *smaller* odd period $\hat{r} + 2$, where $\hat{r} = r - i_1 + i_2$ and $1 < \hat{r} + 2 < r + 2$ such that the iterates $f^j(\hat{w})$, $0 \le j \le \hat{r} + 1$, are jumping alternately around z and $z < \hat{w} < f^{\hat{r}+1}(\hat{w})$. Proceeding in this manner finitely many times, we eventually obtain a periodic point q of odd period $1 < \ell < m$ such that

$$f^{\ell-2}(q) < f^{\ell-4}(q) < \cdots < f^3(q) < f(q) < z < q < f^2(q) < f^4(q) < \cdots < f^{\ell-3}(q) < f^{\ell-1}(q).$$

That is, The orbit of q with respect to f is a Štefan cycle. It is now easy to see that f has, for each odd $n \ge \ell$ (and so for each $n \ge m$), a Štefan cycle of odd period n.

8 The fifth digraph proof of (a), (b) and (c)

We now reconsider how the iterates x_s, $f(x_s)$, $f^2(x_s)$, $\cdots$, $f^{m-1}(x_s)$ "jump" around the fixed point z under the assumption that, when $m \ge 3$ is odd, f has no periodic points of smaller odd periods other than fixed points.

If for all integers k such that $1 \le k \le m-1$ the points $f^k(x_s)$ and $f^{k+1}(x_s)$ lie on opposite sides of z, then, since $f(x_s) > z > x_s$, m is even and, $f^i(x_s) < z < f^j(x_s)$ for all even i and all odd j in $[0, m-1]$. This implies that $f([x_1, x_s]) \supset [x_{s+1}, x_m]$ and $f([x_{s+1}, x_m]) \supset [x_1, x_s]$. Consequently, f has a period-2 point.

On the other hand, assume that there is a *smallest* integer r in $[1, m-1]$ such that the points $f^r(x_s)$ and $f^{r+1}(x_s)$ lie on the same side of z (and so the iterates x_s, $f(x_s)$, $f^2(x_s)$, $\cdots, f^r(x_s)$ are jumping around z alternately and this includes the case when m is odd). The following proof of (a) is not needed here. We include it for the sake of interest in itself. For any finite set A of real numbers, let $H(A)$ denote the interval $[\min A, \max A]$. If r is even, let $I_0 = H(\{x_s, f^2(x_s), f^4(x_s), \cdots, f^r(x_s)\})$ and $I_1 = H(\{z, f(x_s), f^3(x_s), \cdots, f^{r-1}(x_s)\})$. If r is odd, let $I_0 = H(\{x_s, f^2(x_s), f^4(x_s), \cdots, f^{r-1}(x_s),\ z\})$ and $I_1 = H(\{x_{s+1}, f(x_s), f^3(x_s), \cdots, f^r(x_s)\})$ (x_{s+1} is needed only when $r = 1$). In either case, $I_0 \cap I_1 = \emptyset$ and, $f(I_0) \supset I_1$ and $f(I_1) \supset I_0$. So, f has a period-2 point. This proves (a).

For the proofs of (b) and (c), let $m \ge 3$ be odd. If $m = 3$, then f has periodic points of all periods and we are done. So, suppose $m > 3$. Without loss of generality, we may also suppose f has no non-fixed periodic points of smaller odd periods. Since m is odd, there is a *smallest* integer $1 \le r \le m-1$ such that the points $f^r(x_s)$ and $f^{r+1}(x_s)$ lie on the same side of z. If $1 \le r \le m-3$, let $J_i = [z : f^i(x_s)]$ for all $0 \le i \le r-1$, and $J_r = [f^r(x_s), x_s]$ if $f^r(x_s) < x_s$ and $J_r = [x_{s+1}, f^r(x_s)]$ if $x_{s+1} < f^r(x_s)$. By considering the cycle $J_0 J_1 J_2 \cdots J_r([x_s, x_{s+1}])^{m-3-r} J_0$ of length $m-2$, we obtain a non-fixed periodic point of f with odd period $\le m-2$. This is a contradiction. So, for all $1 \le k \le m-3$, the points $f^k(x_s)$ and $f^{k+1}(x_s)$ lie on opposite sides of z. Since $f(x_s) > z$, this implies that $f^i(x_s) < z < f^j(x_s)$ for all even $0 \le i \le m-3$ and all odd $1 \le j \le m-2$. In particular, $z < f^{m-2}(x_s)$. We now have three cases to consider depending on the locations of the point $f^{m-1}(x_s)$:

Case 1. $f^{m-1}(x_s) < z < f^{m-2}(x_s)$. In this case, we actually have $f^i(x_s) < z < f^j(x_s)$ for all even $0 \le i \le m-1$ and all odd $1 \le j \le m-2$. For $0 \le i \le m-1$, let $J_i = [z : f^i(x_s)]$. If, for some $1 \le \ell < k \le m-1$ with $k - \ell$ even, we have $[z : f^k(x_s)] \subset [z : f^\ell(x_s)]$, then by considering the cycle $J_0 J_1 \cdots J_{\ell-1} J_k J_{k+1} \cdots\ J_{m-2}\ [f^{m-1}(x_s), x_s] J_0$ if $k < m-1$ or the cycle $J_0 J_1 J_2 \cdots J_{\ell-1} [f^{m-1}(x_s), x_s] J_0$ if $k = m-1$ of odd length $m - k + \ell$, we obtain a periodic point of f with odd period $m - k + \ell \le m-2$ other than fixed points. This contradicts the assumption. So, we must have

$$f^{m-1}(x_s) < \cdots < f^4(x_s) < f^2(x_s) < x_s < f(x_s) < f^3(x_s) < \cdots < f^{m-2}(x_s).$$

That is, P is a Štefan cycle. Consequently, for each $n > m$, by considering the cycle

$$J_0 J_1 J_2 \cdots J_{m-2} [f^{m-1}(x_s), f^{m-3}(x_s)]([x_s, f(x_s)])^{n-m} J_0$$

of length n, we obtain a period-n point for f. Furthermore, let $U = [f^{m-3}(x_s), z]$, $V = [z, f^{m-2}(x_s)]$ and $W = [f^{m-1}(x_s), f^{m-3}(x_s)]$. Then $f(U) \supset V$, $f(V) \supset W$, and $z \in f(W)$. By Lemma 6, f has periodic points of all even periods ≥ 2.

Case 2. $x_{s+1} = f^{m-1}(x_s) < f^{m-2}(x_s)$. In this case, we have $f^i(x_s) < z < x_{s+1} = f^{m-1}(x_s) < f^j(x_s)$ for all even $0 \leq i \leq m-3$ and all odd $1 \leq j \leq m-2$. Since $x_{s+1} = f^{m-1}(x_s)$, we have $f(x_{s+1}) = x_s$. Thus, $x_s = f(x_{s+1})$. By plugging this in the above inequalities, we obtain that $f^j(x_{s+1}) < z < f^i(x_{s+1})$ for all odd $1 \leq j \leq m-2$ and all even $0 \leq i \leq m-1$. This is a symmetric copy of Case 1. Therefore, P is a Štefan cycle and f has periodic points of all periods $> m$ and of all even periods ≥ 2.

Case 3. $x_{s+1} < f^{m-1}(x_s)$. In this case, since $f(x_{s+1}) \leq x_s < x_{s+1}$ and $f(f^{m-2}(x_s)) = f^{m-1}(x_s) > x_{s+1}$, we have $f^{m-2}(x_s) \neq x_{s+1}$. Hence $x_{s+1} = f^k(x_s)$ for some $1 \leq k \leq m-3$. So, $f^{m-k-2}(x_{s+1}) = f^{m-2}(x_s)$. By considering the cycle

$$[z, x_{s+1}][z : f(x_{s+1})][z : f^2(x_{s+1})] \cdots [z : f^{m-k-3}(x_{s+1})][x_{s+1}, f^{m-2}(x_s)]([x_s, x_{s+1}])^{k-1}[z, x_{s+1}]$$

of length $m-2$, we obtain a periodic point of f with smaller odd period other than fixed points. This contradicts the assumption. So, this case cannot occur.

This shows that P is a Štefan cycle and f has periodic points of all periods $> m$ and of all even periods ≥ 2. Therefore, (a), (b) and (c) are proved.

9 The sixth digraph proof of (a), (b) and (c)

We now consider how points of P which lie on either side of z are mapped.

If for all integers i such that $1 \leq i \leq m-1$ and $i \neq s$ the points $f(x_i)$ and $f(x_{i+1})$ lie on the same side of z, then, since $f(x_{s+1}) < z < f(x_s)$, $f([x_1, x_s] \cap P) \subset [x_{s+1}, x_m] \cap P$ and $f([x_{s+1}, x_m] \cap P) \subset [x_1, x_s] \cap P$. Since f is one-to-one on P, we obtain that $f([x_1, x_s] \cap P) = [x_{s+1}, x_m] \cap P$ and $f([x_{s+1}, x_m] \cap P) = [x_1, x_s] \cap P$ (and so m is even). In particular, $f([x_1, x_s]) \supset [x_{s+1}, x_m]$ and $f([x_{s+1}, x_m]) \supset [x_1, x_s]$. Consequently, f has a period-2 point.

Now *for the rest of this section*, assume that for some integer t such that $1 \leq t \leq m-1$ and $t \neq s$ the points $f(x_t)$ and $f(x_{t+1})$ lie on opposite sides of z (this includes the case when m is odd). Suppose $x_t < x_s$. If $x_{s+1} \leq x_t$, the proof is similar. Since $x_t < x_s$ and $f(x_s) \geq x_{s+1}$, we may also assume that t is the *largest* integer in $[1, s-1]$ such that $f(x_t) \leq x_s$. So, $f([x_t, x_{t+1}]) \supset [x_s, x_{s+1}]$ and $f(x) > z$ for all x in $[x_{t+1}, z] \cap P$.

If $f(x_i) \geq x_{t+1}$ for all $s+1 \leq i \leq m$, then $f^n(x_s) \geq x_{t+1}$ for all $n \geq 1$, contradicting the fact that $f^j(x_s) = x_t$ for some $1 \leq j \leq m-1$. So, there is a *smallest* integer ℓ with $s+1 \leq \ell \leq m$ such that $f(x_\ell) \leq x_t$. If $x_{t+1} \leq f(x_i) \leq x_{\ell-1}$ for all $t+1 \leq i \leq \ell-1$, then $f^n(x_s) \geq x_{t+1}$ for all $n \geq 1$, again contradicting the fact that $f^j(x_s) = x_t$ for some $1 \leq j \leq m-1$. So, there is an integer k with $t+1 \leq k \leq \ell-1$ such that $f(x_k) \geq x_\ell$. If $s+1 < k \leq \ell-1$, by considering the cycles $[x_{s+1}, x_k]([x_k, x_\ell])^n[x_{s+1}, x_k]$, $n \geq 1$, we obtain periodic points of all periods ≥ 2 for f and we are done.

So, *for the rest of this section*, suppose $t+1 \le k \le s$. Let $U = [x_k, z]$, $V = [z, x_\ell]$ and $W = [x_t, x_k]$. Then $f(U) \supset V$, $f(V) \supset W$, and $z \in f(W)$. By Lemma 6, we confirm the existence of periodic points of all even periods, including a period-2 point y of f in $[x_t, x_k]$ such that $x_t < y < x_k < z < f(y) < x_\ell$. This establishes (c) and also (a).

As for the proof of (b), let $m \ge 3$ be odd and let x_t, y, x_k, z, x_ℓ be defined as above. Let r be the *smallest odd* integer in $[2, m]$ such that $f^r(x_k) \le x_k$. For each *odd* integer $n > m$, by considering the cycle

$$[x_t, x_k][z, f(x_k)][z : f^2(x_k)] \cdots [z : f^{r-1}(x_k)]([x_k, z][z, x_\ell])^{(n-r)/2}[x_t, x_k]$$

or the cycle

$$[y, x_k][f(y) : f(x_k)][f^2(y) : f^2(x_k)] \cdots [f^{r-1}(y) : f^{r-1}(x_k)]([x_k, z][z, x_\ell])^{(n-r)/2}[y, x_k]$$

of length n, we obtain (from each cycle) a period-n point of f in $[x_t, x_k]$ whose odd (mod $n+1$) iterates visit the interval $[\min I, x_k]$ ($\supset [x_t, x_k] \supset [y, x_k]$) exactly once. This proves (b).

We now consider iterates of the (different) point x_ℓ : Let $\hat{r}$ be the *smallest even* integer in $[2, m+1]$ such that $f^{\hat{r}}(x_\ell) \le x_k$. For each *odd* integer $n > m$, by considering the cycle

$$[f(x_\ell), y][f^2(x_\ell) : f(y)] \cdots [f^{\hat{r}-1}(x_\ell) : f^{\hat{r}-2}(y)]([x_k, z][z, x_\ell])^{(n-\hat{r}+1)/2}[f(x_\ell), y]$$

of length n, we obtain a period-n point of f in $[f(x_\ell), y]$ ($\subset [\min I, x_k]$) whose *odd* iterates (mod $n+1$) visit the interval $[\min I, x_k]$ exactly once. This proves (b). Note that the period-n points obtained from the above three cycles may generate three distinct orbits, for example, if $m = 7$, $f^7(x_k) = x_k$ and

$$x_t = f^6(x_k) < y < f^2(x_k) < x_k < z < f(y) < f^5(x_k) = x_\ell < f^3(x_k) < f(x_k) < f^4(x_k).$$

When $m \ge 5$ is odd, our method can even be used to prove the existence of Štefan cycles of all odd periods $n \ge m$ *without assuming the non-existence of periodic points of odd periods* $< m$ *other than fixed points*. Indeed, as before, let t be the largest integer in $[1, s-1]$ such that $f(x_t) \le x_s$. Then $f(x_i) > z$ for all $t+1 \le i \le s$. If $f(x_i) < z$ for all $s+1 \le i \le m$, then there is a *smallest* integer ℓ in $[s+1, m]$ such that $f(x_\ell) \le x_t$. On the other hand, if there is also a *smallest* integer r in $(s+1, m]$ such that $f(x_r) > z$, then since $x_t \notin [x_{t+1}, x_{r-1}]$, we have $f(P \cap [x_{t+1}, x_{r-1}]) \not\subset [x_{t+1}, x_{r-1}]$. So, for some ℓ in $[t+1, r-1]$, $f(x_\ell) \notin [x_{t+1}, x_{r-1}]$. Thus, we have either $t+1 \le \ell \le s$ and $f(x_\ell) \ge x_r$ or, $s+1 \le \ell < r$ and $f(x_\ell) \le x_t$. Since the proofs for these two cases and the above one are similar, we may assume that there is a *smallest* integer ℓ in $[s+1, m]$ such that $f(x_\ell) \le x_t$ and, $f(x_i) > z$ for all $t+1 \le i \le s$ and $f(x_j) < z$ for all $s+1 \le j \le \ell-1$. Consequently, $f(x_k) \ge x_\ell$ for some $t+1 \le k \le s$.

Since $f(x_k) \ge x_\ell$, the point $u_0 = \max\{x_k \le x < z : f(x) = x_\ell\}$ exists and $f(x) < x_\ell$ for all x in (u_0, z). If $P \cap (u_0, z) = \emptyset$, then $x_s \le u_0$. In this case, let $j = 0$. If $P \cap (u_0, z) \ne \emptyset$, let p be a point in $P \cap (u_0, z)$. Then $z < f(p) \le x_{\ell-1}$. It is clear that $f(p_{-1}) \le u_0$ for some p_{-1} in $P \cap (z, x_{\ell-1}]$. So, the point $u_{-1} = \min\{z < x \le p_{-1} : f(x) = u_0\}$ exists and $u_0 < f(x)$ for all x in (z, u_{-1}). Since $p \in [u_0, z]$, it is also clear that $f(p_{-2}) \ge u_{-1}$ for some p_{-2} in

$P \cap (u_0, z)$. So, the point $u_{-2} = \max\{p_{-2} \le x < z : f(x) = u_{-1}\}$ exists and $f(x) < u_{-1}$ for all x in (u_{-2}, z). If $P \cap (u_{-2}, z) = \emptyset$, then $x_s \le u_{-2}$. Otherwise let $\tilde{p}$ be a point in $P \cap (u_{-2}, z)$. Then $z < f(\tilde{p}) < u_{-1}$. We proceed in this manner. Since (x_k, x_ℓ) contains at most $m-3$ points of P (exclude at least these three points x_t, x_k, x_ℓ), there is an *even* integer $0 \le j \le m-3$ and points $p_{-1}, p_{-2}, \cdots, p_{-j}$ in P and points u_0, u_{-1}, $u_{-2}, \cdots, u_{-j}$ in $[x_k, x_\ell]$ such that $P \cap (u_{-j}, z) = \emptyset$, $f(u_{-i}) = u_{-i+1}$ for all $1 \le i \le j$, and

$$x_k \le u_0 < p_{-2} \le u_{-2} < \cdots < p_{-j+2} \le u_{-j+2} < p_{-j} \le u_{-j} < z$$
$$< u_{-j+1} \le p_{-j+1} < u_{-j+3} \le p_{-j+3} < \cdots < u_{-1} \le p_{-1} < x_\ell.$$

It is clear that there exist more points $u_{-\hat{i}}$, $\hat{i} > j$, such that $f(u_{-\hat{i}}) = u_{-\hat{i}+1}$, $\hat{i} > j$ and

$$u_{-j} < u_{-j-2} < u_{-j-4} < \cdots < z < \cdots < u_{-j-3} < u_{-j-1} < u_{-j+1}.$$

Since $P \cap (u_{-j}, z) = \emptyset$, we have $x_s \le u_{-j}$. Since $f(x_t) \le x_s$,we have $f([x_t, x_k]) \supset [x_s, x_\ell] \supset [x_{-j}, z]$. For each *even* integer $n \ge j$, by considering the cycle

$$[x_t, x_k][u_{-n}, z][z, u_{-n+1}][u_{-n+2}, u_{-n}][u_{-n+1}, u_{-n+3}] \cdots [u_0, u_{-2}][u_{-1}, x_\ell][x_t, x_k]$$

of length $n+3$ $(\ge j+3)$, we obtain a Štefan cycle of least period $n+3$. Since j is even and $0 \le j \le m-3$, this shows that f has a Štefan cycle of least period n for each odd $n \ge m$.

If f has no periodic points of odd periods in $[3, m-2]$, then j must be equal to $m-3$ and there is eaxtly one point of P in each of the $m-3$ half-open intervals $(u_0, u_{-2}]$, $(u_{-2}, u_{-4}]$, $\cdots, (u_{-j+2}, u_{-j}]$, $[u_{-j+1}, u_{-j+3})$, $\cdots, [u_{-5}, u_{-3})$, $[u_{-3}, u_{-1})$, $[u_{-1}, x_\ell)$. These $m-3$ points, plus x_k, x_ℓ, and x_t, constitute the orbit P. Therefore, P is itself a Štefan cycle.

10 The seventh digraph proof of (a), (b) and (c)

We now reconsider (cf. **[14, 16]**) how the points of P which lie on either side of the fixed point z are mapped *under the assumption that, when $m \ge 5$ is odd, f has no periodic points of smaller odd periods other than fixed points.*

If for all integers i such that $1 \le i \le m-1$ and $i \ne s$ the points $f(x_i)$ and $f(x_{i+1})$ lie on the same side of z, then it is easy to see that $f([x_1, x_s]) \supset [x_{s+1}, x_m]$ and $f([x_{s+1}, x_m]) \supset [x_1, x_s]$. So, f has a period-2 point. On the other hand, assume that there is an integer $1 \le t \le m-1$ such that $t \ne s$ and the points $f(x_t)$ and $f(x_{t+1})$ lie on opposite sides of z (this includes the case when $m \ge 3$ is odd). Without loss of generality, we may assume that $x_t < x_s$. We may also assume that t is the *largest* integer in $[1, s)$ such that $f(x_t) \le x_s$. So, $f(x_i) \ge x_{s+1}$ for all $t+1 \le i \le s$. Now let q be the *smallest* positive integer such that $f^q(x_s) \le x_t$. Then $1 \le q \le m-1$. By considering the cycles

$$[x_s, x_{s+1}][z, f(x_s)][z : f^2(x_s)] \cdots [z : f^{q-1}(x_s)][x_t, x_{t+1}]([x_s, x_{s+1}])^{i+1}, \; i \ge 0,$$

we obtain that f has periodic points of all periods $\ge q+i+1$, $i \ge 0$. In particular, when $m \ge 3$ is odd, this establishes (b).

For the proof of (c), we may assume that $m \ge 5$ is odd and f has no periodic points of smaller odd periods other than fixed points. Under this assumption, we want to show that

$$f^{m-1}(x_s) = x_t < x_{t+1} \le f^{m-3}(x_s) < z < f^{m-2}(x_s).$$

Then by Lemma 6, we easily obtain (c) and (a). Since we have shown in the above that f has periodic points of least periods $q+i+1$, $i \ge 0$, we see that q cannot be $\le m-3$. So, either $q = m-2$ or $q = m-1$, and $x_{t+1} \le f^i(x_s)$ for all $0 \le i \le m-3$. In the following, we present three different arguments.

If $f^{m-2}(x_s) \le x_t$, then it is easy to see that $f^{m-1}(x_s)$ cannot be $\ge x_{t+1}$. So, we also have $f^{m-1}(x_s) \le x_t$. But then $x_{s+1} = f^k(x_s)$ for some $1 \le k \le m-3$. By considering the cycle

$$[x_s, f^k(x_s)][x_s : f^{k+1}(x_s)][x_s : f^{k+2}(x_s)] \cdots [x_s : f^{m-3}(x_s)][f^{m-2}(x_s), x_{t+1}]([x_s, x_{s+1}])^k$$

of length $m-2$, we obtain a periodic point of samller odd period other than fixed point which is a contradiction. Therefore, $f^{m-1}(x_s) \le x_t < f^{m-2}(x_s)$. Consequently, we have $f^{m-1}(x_s) = x_t$ and $z < f^{m-2}(x_s)$. If $f^{m-3}(x_s) > z$, then by considering the cycle

$$[x_s, x_{s+1}][x_s : f(x_s)][x_s : f^2(x_s)] \cdots [x_s : f^{m-4}(x_s)][x_{s+1}, f^{m-3}(x_s)][x_s, x_{s+1}]$$

of length $m-2$, we reach a contradiction. Thus, $f^{m-1}(x_s) = x_t < x_{t+1} \le f^{m-3}(x_s) < z < f^{m-2}(x_s)$.

Now we present the second argument. It is clear that there is a *smallest* integer $1 \le r \le m-1$ such that both $f^r(x_s)$ and $f^{r+1}(x_s)$ lie on the same side of z. If $r = 1$, then we consider the cycle $[x_s, x_{s+1}][x_{s+1}, f(x_s)]([x_s, x_{s+1}])^2$. If $1 < r \le m-3$, then we consider the cycle

$$[x_s, x_{s+1}][z : f(x_s)][x_s : f^2(x_s)][f(x_s) : f^3(x_s)] \cdots [f^{r-2}(x_s) : f^r(x_s)]([x_s, x_{s+1}])^{m-2-r}$$

of length $m-2$. In either case, we have a periodic point of smaller odd period other than fixed point which is a contradiction. So, $r \ge m-2$. In particular, $(x_{t+1} \le)\, f^i(x_s) < z < f^j(x_s)$ for all even $0 \le i \le m-3$ and all odd $1 \le j \le m-2$. This forces $f^{m-1}(x_s) = x_t$ and we have $f^{m-1}(x_s) = x_t < x_{t+1} \le f^{m-3}(x_s) < z < f^{m-2}(x_s)$. That P is actually a Štefan cycle can be argued as that in Case 1 of section 8.

Here is the third argument. If $f(x_{\hat{t}}) \ge x_{s+1}$ for some $s+1 < \hat{t} \le m$, then it is clear that $x_{\hat{t}} \ne f^{m-2}(x_s), f^{m-1}(x_s)$. So, $x_{\hat{t}} = f^r(x_s)$ for some $1 \le r \le m-3$. If $r = 1$, we consider the cycle $[x_s, x_{s+1}]\ [x_{s+1}, f(x_s)]\ ([x_s, x_{s+1}])^2$. Otherwise, we consider the cycle

$$[x_s, x_{s+1}][z : f(x_s)][z : f^2(x_s)] \cdots [z : f^{r-1}(x_s)][x_{s+1}, f^r(x_s)]([x_s, x_{s+1}])^{m-2-r}$$

of length $m-2$. In either case, we have a periodic point of smaller odd period other than fixed point which is a contradiction. Therefore, f maps all x_i, $s+1 \le i \le m$, to the left side of z. Since f maps all x_j, $t+1 \le j \le s$ to the right side of z and since $x_{t+1} \le f^i(x_s)$ for all $0 \le i \le m-3$, we obtain that $x_{t+1} \le f^{i_1}(x_s) < z < f^{i_2}(x_s)$ for all even $0 \le i_1 \le m-3$ and all odd $1 \le i_2 \le m-2$. Thus $q = m-1$ and $f^{m-1}(x_s) = x_t$. In particular, $f^{m-1}(x_s) = x_t < x_{t+1} \le f^{m-3}(x_s) < z < f^{m-2}(x_s)$.

For completeness, we include a proof of Sharkovsky's theorem in which the proof of (1) adopts the strategy of section 7 and is different from those in **[14, 15]**. It is self-contained and does not refer to (a), (b) or (c). We also introduce three new doubling operators and use one of them as an example to construct new examples for (2) and (3).

11 A proof of Sharkovsky's theorem

If f has a period-m point with $m \ge 2$, then it is clear that f has a fixed point. If f has a period-m point with $m \ge 3$ and odd, let $\{x_1, x_2, \cdots, x_m\}$ be a period-m orbit of f with $x_1 < x_2 < \cdots < x_m$. Let $1 \le s \le m-1$ be an integer such that $f(x_s) \ge x_{s+1}$ and $f(x_{s+1}) \le x_s$. Let z be a fixed point of f in $[x_s, x_{s+1}]$. Since m is odd, there is a *smallest* integer $1 \le r \le m-1$ such that both $f^r(x_s)$ and $f^{r+1}(x_s)$ lie on the same side of z. If $r = 1$, we consider the cycles $[x_s, x_{s+1}][x_{s+1}, f(x_s)]([x_s, x_{s+1}])^i$, $i \ge 1$. If $r = 2$, we consider the cycles $[x_s, x_{s+1}]\ [z, f(x_s)]\ [f^2(x_s), x_s]\ ([x_s, x_{s+1}])^i$, $i \ge 1$. In either case, we have periodic points of all periods ≥ 3, including period-$(m+2)$. If $r \ge 3$, let $L_i = [f^i(x_s) : f^{i+2}(x_s)]$ for $0 \le i \le r-3$. For each integer $n \ge m+1$, we consider the cycle

$$[x_s, x_{s+1}][z, f(x_s)]L_0L_1L_2L_3 \cdots L_{r-3}[f^{r-2}(x_s) : f^r(x_s)]([x_s, x_{s+1}])^{n-r}$$

of length n to obtain a period-n point p_n of f in $[x_s, x_{s+1}]$ such that $f^i(p_n) \notin [x_s, x_{s+1}]$ for all $2 \le i \le r$ and $f^j(p_n) \in [x_s, x_{s+1}]$ for all $r+1 \le j \le n$. In particular, this shows that if f has a period-m point with $m \ge 3$ and odd, then f has a period-$(m+2)$ point. That is, (b) holds.

On the other hand, let $1 \le k \le r\ (\le m-1)$ be the smallest integer such that $[z : f^k(x_s)] \supset [z : f^r(x_s)]$. If $k = 1$, then by considering the cycles $[x_s, x_{s+1}][x_{s+1}, f(x_s)]([x_s, x_{s+1}])^i$, $i \ge 1$, we have periodic points of all periods ≥ 2 and we are done. So, suppose $k \ge 2$. Then $[z : f^{k-2}(x_s)] \subset [z : f^r(x_s)]$. By considering the cycles

$$[f^{k-2}(x_s)] : f^r(x_s)][z : f^{k-1}(x_s)]\ ([z : f^{k-2}(x_s)][z : f^{k-1}(x_s)])^i\ [f^{k-2}(x_s)] : f^r(x_s)],\ i \ge 0,$$

we obtain periodic points of all even periods ≥ 2, including period-6 and period-$(2m)$ for f. That is, (c) holds.

If f has a period-$(2 \cdot m)$ point with $m \ge 3$ and odd, then f^2 has a period-m point. It follows from the above that (or by (b) and (c)) f^2 has a period-$(m+2)$ point and a period-6 point. If f^2 has a period-$(m+2)$ point, then f has either a period-$(m+2)$ point or a period-$(2 \cdot (m+2))$ point. If f has a period-$(m+2)$ point, then it follows again from the above that (or by (b)) f has a period-$(2 \cdot (m+2))$ point. In either case, f has a period-$(2 \cdot (m+2))$ point. On the other hand, if f^2 has a period-$(2 \cdot 3)$ point, then, by Lemma 3(2), f has a period-$(2^2 \cdot 3)$ point. This shows that if f has a period-$(2 \cdot m)$ point with $m \ge 3$ and odd, then f has a period-$(2 \cdot (m+2))$ point and a period-$(2^2 \cdot 3)$ point.

Now if f has a period-$(2^k \cdot m)$ point with $m \ge 3$ and odd and if $k \ge 2$, then $f^{2^{k-1}}$ has a period-$(2 \cdot m)$ point. It follows from the previous paragraph that $f^{2^{k-1}}$ has a period-$(2 \cdot (m+2))$ point and a period-$(2^2 \cdot 3)$ point. So, by Lemma 3(2), f has a period-$(2^k \cdot (m+2))$ point and a period-$(2^{k+1} \cdot 3)$ point. Furthermore, if f has a period-$(2^i \cdot m)$ point with $m \ge 3$ and odd and if $i \ge 0$, then f^{2^i} has a period-m point. For each $\ell \ge i$, by Lemma 3(1), $f^{2^\ell} = (f^{2^i})^{2^{\ell-i}}$ has a period-m point and so f^{2^ℓ} has a period-6 point. Thus, $f^{2^{\ell+1}}$ has a period-3 point and hence, has a period-2 point. This implies that f has a period-$2^{\ell+2}$ point for each $\ell \ge i$.

Finally, if f has a period-2^k point for some $k \ge 2$, then $f^{2^{k-2}}$ has a period-4 point. Let $g = f^{2^{k-2}}$ and let $\{x_1, x_2, x_3, x_4\}$ be a period-4 orbit of g with $x_1 < x_2 < x_3 < x_4$.

Let $1 \leq s \leq 3$ be an integer such that $g(x_s) \geq x_{s+1}$ and $g(x_{s+1}) \leq x_s$. Let $\hat{z}$ be a fixed point of g in $[x_s, x_{s+1}]$. If $g^i(x_s)$ and $g^{i+1}(x_s)$ lie on opposite sides of $\hat{z}$ for $i = 1, 2$, then $g^2(x_s) < x_s < \hat{z} < \min\{g(x_s), g^3(x_s)\}$. By considering the cycle $[g^2(x_s), x_s][g(x_s) : g^3(x_s)][g^2(x_s), x_s]$, we obtain a period-2 point of g. If $\hat{z} < \min\{g(x_s), g^2(x_s)\}$, then we consider the cycle $[x_s, x_{s+1}][x_{s+1}, g(x_s)][x_s, x_{s+1}]$. If $\max\{g^2(x_s), g^3(x_s)\} < \hat{z}$, then we consider the cycle $[g^2(x_s), x_s][x_s, g(x_s)][g^2(x_s), x_s]$. In either case, we obtain a period-2 point of $g = f^{2^{k-2}}$. Therefore, f has a period-2^{k-1} point and hence, by induction, has a period-2^j point for each $j = 1, 2, \cdots, k-2$. This establishes (1).

As for the proofs of (2) and (3), it suffices to consider the tent map $T(x) = 1 - |2x - 1|$ defined on $I = [0, 1]$ which, for each integer $n \geq 1$, has finitely many period-n orbits and each period-n orbit Q_n with $n \geq 2$ must satisfy $T(\max Q_n) = \min Q_n$. Among these period-n orbits, let P_n be one such that the open interval $(\min P_n, \max P_n)$ contains no period-n orbits of T. For any x in I, let $T_n(x) = T(x)$ if $0 \leq x \leq \max P_n$ and $T_n(x) = \min P_n$ if $\max P_n \leq x \leq 1$. Then it is easy to see that T_n (which looks different from that in **[14, 15]**) has exactly one period-n orbit (i.e., P_n) but has no period-m orbit for any m with $n \prec m$ in the Sharkovsky ordering. This and the constant maps confirm (2). (3) is left to the readers.

In the following, we shall construct examples for (2) and (3) by introducing three new doubling operators which are different from the classical one **[1, 17, 32]**. The classical doubling operator can be loosely described as pushing up and pulling back. Ours go one or two steps further. In the following, we introduce a new doubling operator which can be loosely described as pushing up with an upside-down turn and pulling back with an upside-down turn. We shall introduce the other two in the next section. Let a be a fixed number in $(0, 1/2)$. For any continuous map f from $[0, a]$ into itself, we define the doubling operator F_a (see Figure 1) of f to be the continuous map from $[0, 1]$ into itself defined by

$$(F_a(f))(x) = \begin{cases} 1 - f(x), & 0 \leq x \leq a, \\ \text{decreasing on } [a, 1-a], & a \leq x \leq 1-a, \\ 1 - x, & 1-a \leq x \leq 1. \end{cases}$$

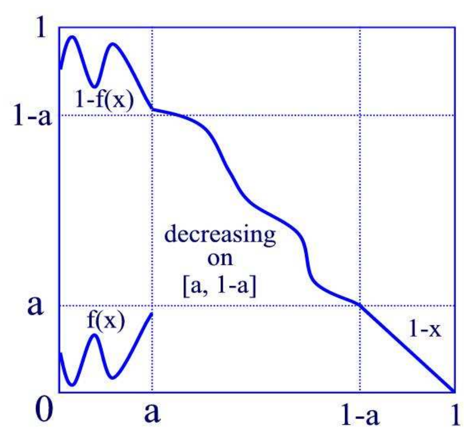

Figure 1: The graph of the doubling operator $F_a(f)$ of the map f on $[0, a]$.

When there is no ambiguity, we'll simply write F instead of $F_a(f)$. It is clear that $F([0,a]) \subset [1-a,1]$ and $F([1-a,1]) \subset [0,a]$ and $F^2(x) = f(x)$ on $[0,a]$. Furthermore, since F is decreasing on $[a, 1-a]$, F can only have a fixed point, and maybe some period-2 points in $[a, 1-a]$. However, F already has period-2 points in $[0,a]$, i.e., the fixed points of f. Therefore, we obtain that $\{m : F$ has a period-m point in $[0,1]\} = \{2n : f$ has a period-n point in $[0,a]\} \cup \{1\}$. Note that the action of F on $[0,a]$ is pushing up $1-a$ units and turning upside down along the center line $y = 1 - a/2$ and, on $[1-a, 1]$ is pulling down $1-a$ units and turning upside down along the center line $y = a/2$. Since topological conjugacy preserves the periods of periodic points, we can use F to define a doubling operator for any continuous map from $[0,1]$ (instead of $[0,a]$) into itself. For any continuous map g from $[0,1]$ into itself, the topologically conjugate map $ag(x/a)$ is a continuous map from $[0,a]$ into itself. So, we can let $G_a(g)(x) = F_a(ag(x/a))$. Then $\{m : G_a(g)$ has a period-m point in $[0,1]\} = \{2n : g$ has a period-n point in $[0,1]\} \cup \{1\}$. Therefore, G_a doubles the periods of all periodic points of any continuous map from $[0,1]$ into itself.

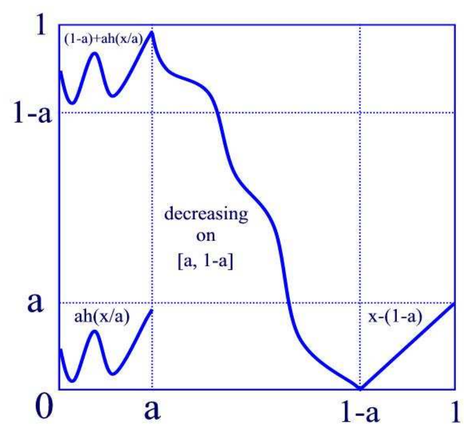

Figure 2: The graph of the classical doubling operator $H_a(h)$ of the map h on $[0,1]$.

For convenience, we also include the classical doubling operator introduced in **[1, 32]** as follows: For any continuous map h from $[0,1]$ into itself, we let $H_a(h)$ (see Figure 2) to be the continuous map from $[0,1]$ into itself defined by

$$(H_a(h))(x) = \begin{cases} ah(x/a) + (1-a), & 0 \le x \le a, \\ \text{decreasing on } [a, 1-a], & a \le x \le 1-a, \\ x - (1-a), & 1-a \le x \le 1. \end{cases}$$

Then $\{m : H_a(h)$ has a period-m point in $[0,1]\} = \{2n : h$ has a period-n point in $[0,1]\}$ $\cup\{1\}$. Therefore, H_a also doubles the periods of all periodic points of any continuous map from $[0,1]$ into itself. Consequently, we now have two different doubling operators, i.e., G_a and H_a. We shall use these two operators later on.

For every integer $n \ge 2$, let $x_i, 1 \le i \le 2n+1$, be $2n+1$ distinct points in $[0,1]$ such that $0 = x_1 < x_2 < \cdots < x_{2n} < x_{2n+1} = 1$. Let $f_n(x)$ be the continuous map from $[x_1, x_{2n+1}] (= [0,1])$ onto itself defined by putting $f_n(x_1) = x_{n+1}$; $f_n(x_i) = x_{2n+3-i}$

for $2 \le i \le n+1$; $f_n(x_j) = x_{2n+2-j}$ for $n+2 \le j \le 2n+1$; and by linearity on each interval $[x_k, x_{k+1}]$, $1 \le k \le 2n$. Then it is easy to see that the set $\{x_i : 1 \le i \le 2n+1\}$ forms a Štefan cycle of f_n with least period $2n+1$ and $f_n^{2n-1}([x_1, x_2]) = [x_2, x_{2n+1}]$. So, $f_n^{2n-1}([x_1, x_2]) \cap [x_1, x_2) = \emptyset$. That is, the interval $[x_1, x_2]$ contains no period-$(2n-1)$ point of f_n. Furthermore, since f_n is strictly decreasing on $[x_2, x_{2n+1}]$, any periodic orbit of least period > 2 must have at least one point in $[x_1, x_2]$ and hence f_n cannot have period-$(2n-1)$ points. Therefore, f_n has period-$(2n+1)$ points but no period-$(2n-1)$ points. On the other hand, on $[0,1]$, let $g(x) \equiv 0$ be the constant map and let $h(x) = 1/2 + 2x$ for $0 \le x \le 1/4$; $h(x) = -2x + 3/2$ for $1/4 \le x \le 1/2$; and $h(x) = 1 - x$ for $1/2 \le x \le 1$. Then g has fixed points but no period-2 points and h has period-6 points (for example, the point $x = 1/2 - 1/7$) but no periodic points of any odd periods > 1. By applying successively the doubling operator G_a or H_b (with any choices of a and b in $(0, 1/2)$) to f_n, g, and h respectively, we obtain (2).

Finally, as for examples for (3), let $< a_i >_{i \ge 1}$, $< b_i >_{i \ge 1}$ be any two infinite sequences of numbers in $(0, 1/2)$ and let $< \alpha_i >_{i \ge 1}$ be any infinite sequence of 0's and 1's. For $i \ge 1$, let

$$\Phi_{\alpha_i} = \begin{cases} G_{a_i}, & \text{if } \alpha_i = 0, \\ H_{b_i}, & \text{if } \alpha_i = 1. \end{cases} \qquad \text{and} \qquad c_i = \begin{cases} a_i, & \text{if } \alpha_i = 0, \\ b_i, & \text{if } \alpha_i = 1. \end{cases}$$

Then, for any fixed $n > 2$ and any continuous map ϕ from $[0,1]$ into itself, since we have $(\Phi_{\alpha_n}(\Phi_{\alpha_{n+1}}(\phi)))(x) = (\Phi_{\alpha_n}(\phi))(x)$ on $[1 - c_n, 1]$, it is easy to see that

$$(\Phi_{\alpha_{n-1}}(\Phi_{\alpha_n}(\Phi_{\alpha_{n+1}}(\phi))))(x) = (\Phi_{\alpha_{n-1}}(\Phi_{\alpha_n}(\phi)))(x) \quad \text{on} \quad [c_{n-1}(1 - c_n), 1],$$

and

$$|(\Phi_{\alpha_{n-1}}(\Phi_{\alpha_n}(\Phi_{\alpha_{n+1}}(\phi))))(x) - (\Phi_{\alpha_{n-1}}(\Phi_{\alpha_n}(\phi)))(x)| < c_{n-1} \quad \text{on} \quad [0, c_{n-1}(1 - c_n)].$$

By induction, we obtain that, on $[c_1 c_2 c_3 \cdots c_{n-2} c_{n-1}(1 - c_n), 1]$,

$$(\Phi_{\alpha_1}(\Phi_{\alpha_2}(\cdots(\Phi_{\alpha_n}(\Phi_{\alpha_{n+1}}(\phi))\cdots)(x) = (\Phi_{\alpha_1}(\Phi_{\alpha_2}(\cdots(\Phi_{\alpha_n}(\phi))\cdots)(x),$$

and, on $[0, c_1 c_2 c_3 \cdots c_{n-2} c_{n-1}(1 - c_n)]$,

$$|(\Phi_{\alpha_1}(\Phi_{\alpha_2}(\cdots(\Phi_{\alpha_n}(\Phi_{\alpha_{n+1}}(\phi))\cdots)(x) - (\Phi_{\alpha_1}(\Phi_{\alpha_2}(\cdots(\Phi_{\alpha_n}(\phi))\cdots)(x)| < \Pi_{i=1}^{n-1} c_i < 1/2^{n-1}.$$

Thus, the sequence $\Phi_{\alpha_1}(\phi)$, $\Phi_{\alpha_1}(\Phi_{\alpha_2}(\phi))$, $\Phi_{\alpha_1}(\Phi_{\alpha_2}(\Phi_{\alpha_3}(\phi)))$, $\cdots$, converges uniformly to a continuous map Φ_α, where $\alpha = \alpha_1 \alpha_2 \cdots$, on $[0,1]$ which is *independent* of ϕ. It is easy to see that Φ_α is an example for (3). Since there are uncountably many α's, we have uncountably many examples Φ_α for (3). Figure 3 is such an example with (i) $a_i = b_i = 1/3$, $i \ge 1$; (ii) $\beta = \beta_1 \beta_2 \beta_3 \cdots = \overline{01} = 010101\cdots$; and (iii) both G_{a_i} and H_{b_i}, $i \ge 1$, are linear on $[1/3, 2/3]$.

Remark. In the above construction of Φ_α with $a_i = b_i = 1/3$, $i \ge 1$, if $\alpha = \overline{1} = 111\cdots$, then we have the classical example as introduced in [**1**] with $\Phi_{\overline{1}}(0) = 1$. On the other hand, for any $\alpha \ne \overline{1}$, there is a $k \ge 1$ such that $\alpha_k = 0$. Then because of the turning upside down property of $G_{a_k} (= \Phi_{\alpha_k})$, we obtain that $0 < \Phi_\alpha(0) < 1$. For example, for $\beta = \overline{01}$ considered above, we have $\Phi_\beta(0) = 7/10$. Similarly, if $a_i \equiv c$, $i \ge 1$, where c is a constant in $(0, 1/2)$, and if $\gamma = \overline{0}$, then we have $\Phi_\gamma(0) = 1/(1+c) \in (0,1)$.

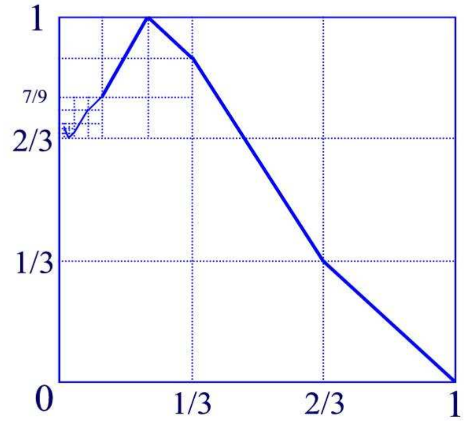

Figure 3: The graph of the map Φ_β. Note that $\Phi_\beta(0) = 7/10$.

12 Two more new doubling operators

In this section, we introduce two more new doubling operators which, together with the classical one and the one in section 10, can be used in various combinations, as in the previous section, to construct new examples for (2) and (3). We shall leave the details to the readers.

Let a be a fixed number in $(0, 1/2)$. For any continuous map f from $[0, a]$ into itself, we define the doubling operator $D = D_a$ (see Figure 4) of f to be the continuous map from $[0, 1]$ into itself defined by

$$(D(f))(x) = \begin{cases} 1 - a + f(a - x), & 0 \le x \le a, \\ \text{decreasing on } [a, 1-a], & a \le x \le 1 - a, \\ 1 - x, & 1 - a \le x \le 1. \end{cases}$$

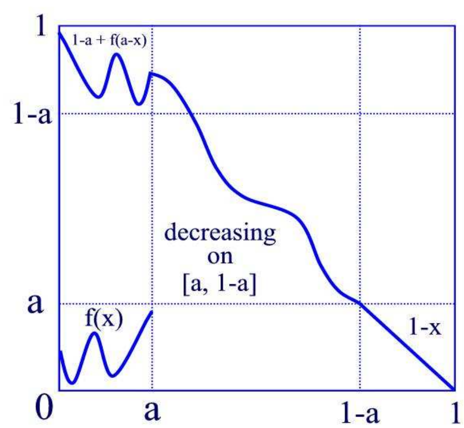

Figure 4: The graph of the doubling operator $D_a(f)$ of the map f on $[0, a]$.

It is clear that $D([0, a]) \subset [1 - a, 1]$ and $D([1 - a, 1]) \subset [0, a]$ and $D^{2n}(x) - x = (a - x) - f^n(a - x)$ on $[0, a]$. So, $x_0 \in [0, a]$ is a period-$(2n)$ point of D if and only if $a - x_0$ is

a period-n point of f. Arguing as before, we obtain that $\{m : D$ has a period-m point in $[0,1]\} = \{2n : f$ has a period-n point in $[0,a]\} \cup \{1\}$. Note that the action of D on $[0,a]$ is pushing up $1-a$ units and interchanging left and right symmetrically of the line $x = a/2$ and, on $[1-a, 1]$ is pulling down $1-a$ units and interchanging left and right symmetrically of the line $x = 1-a/2$. For any continuous map g from $[0,1]$ into itself, the topologically conjugate map $ag(x/a)$ is a continuous map from $[0,a]$ into itself. So, we can let $\hat{D}(g)(x) = D(ag(x/a))$. Then $\hat{D}$ doubles the periods of all periodic points of any continuous map from $[0,1]$ into itself.

For any continuous map f from $[0,a]$ into itself, we define the doubling operator $E = E_a$ (see Figure 5) of f to be the continuous map from $[0,1]$ into itself defined by

$$(E(f))(x) = \begin{cases} 1 - f(a-x), & 0 \le x \le a, \\ \text{decreasing on } [a, 1-a], & a \le x \le 1-a, \\ x - (1-a), & 1-a \le x \le 1. \end{cases}$$

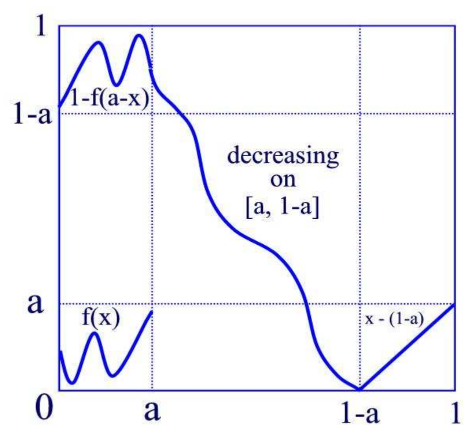

Figure 5: The graph of the doubling operator $E_a(f)$ of the map f on $[0,a]$.

It is clear that $E([0,a]) \subset [1-a, 1]$ and $E([1-a,1]) \subset [0,a]$ and $E^{2n}(x) - x = (a - x) - f^n(a-x)$ on $[0,a]$. So, $x_0 \in [0,a]$ is a period-$(2n)$ point of E if and only if $a - x_0$ is a period-n point of f. Arguing as before, we obtain that $\{m : E$ has a period-m point in $[0,1]\} = \{2n : f$ has a period-n point in $[0,a]\} \cup \{1\}$. Note that the action of E on $[0,a]$ is pushing up $1-a$ units and interchanging left and right symmetrically of the line $x = a/2$ and then turning upside down along the center line $y = 1 - a/2$ and, on $[1-a, 1]$ is pulling down $1-a$ units and interchanging left and right symmetrically of the line $x = 1-a/2$ and then turning upside down along the center line $y = a/2$. For any continuous map g from $[0,1]$ into itself, by letting $\hat{E}(g)(x) = E(ag(x/a))$, we see that $\hat{E}$ doubles the periods of all periodic points of any continuous map from $[0,1]$ into itself.

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